GCSE Maths – Algebra

Completing the Square (Higher Only)

Notes

WORKSHEET

Completing the Square

Completing the square is a method which can be used to calculate the **turning point, line of symmetry** and **solutions** of a quadratic equation. Completing the square manipulates a quadratic so that it is in the form

$$y = a(x + d)^2 + e.$$

Complete the square

This method is a way of rearranging a quadratic formula, so it is easier to make x the **subject** of the equation. Completing the square of the quadratic equation

$$x^2 + bx + c = 0$$

can be done using the formula

$$\left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = 0.$$

The following steps illustrate how the second equation above is obtained:

1. We start with an equation of the form

$$x^2 + bx + c = 0$$

2. Split the x term in half and write it as two separate terms:

$$x^2 + \frac{b}{2}x + \frac{b}{2}x + c = 0$$

3. Factorise the x terms into double brackets:

$$x^2 + \frac{b}{2}x + \frac{b}{2}x = \left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = \left(x + \frac{b}{2}\right)^2$$

4. When the double brackets are expanded this results in:

$$\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = x^2 + bx + \left(\frac{b}{2}\right)^2$$

5. Comparing this with the equation we started with, we need to subtract the last term $\left(\frac{b}{2}\right)^2$ and add the constant term c:

$$\left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = 0$$

Example: Complete the square of $x^2 - 8x + 17$

1. Compare with the general form $x^2 + bx + c$ introduced above and label the coefficients:

$$b = -8, \qquad c = 17$$

2. Divide the b coefficient by 2:

$$\frac{b}{2} = -8 \div 2 = -4$$

3. Substitute the values for b and c into the general formula.

$$\left(\boldsymbol{x} + \frac{\boldsymbol{b}}{\mathbf{2}}\right)^2 + \boldsymbol{c} - \left(\frac{\boldsymbol{b}}{\mathbf{2}}\right)^2$$

$$(x-4)^2 + 17 - (-4)^2$$

4. Simplify the expression.

$$(x-4)^2 + 17 - (-4)^2 = \boldsymbol{(x-4)^2 + 1}$$

So, the completed square form is $\boldsymbol{(x-4)^2 + 1}$.

When the x^2 term has a coefficient a in front, completing the square of

$$\boldsymbol{ax^2 + bx + c = 0}$$

gives a completed square form

$$\boldsymbol{a(x+p)^2 + q}.$$

To complete the square, **factorise the equation** by a.

Example: Complete the square of $2x^2 + 8x + 10$

1. Factorise the equation by the coefficient of x^2:

$$2(x^2 + 4x + 5)$$

2. Now we can complete the square of the expression in the brackets. Comparing with the expression $x^2 + bx + c$, label the coefficients:

$$b = 4, \qquad c = 5$$

3. Substitute values into the general equation to complete the square:

$$\left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = (x+2)^2 + 5 - 4$$

4. Replace the expression in the brackets and expand out the coefficient which was factorised in Step 1.

$$2[(x+2)^2 + 1] = \boldsymbol{2(x+2)^2 + 2}$$

Finding turning point and the line of symmetry

Completing the square can be used to find the **turning point** and **line of symmetry** of a quadratic equation. Once having undergone completing the square, a quadratic equation can be written as

$$\boldsymbol{a(x+p)^2+q=0}.$$

- The turning point of a quadratic equation is $(-\boldsymbol{p}, \boldsymbol{q})$.
- The line of symmetry is $\boldsymbol{x=-p}$.

Example: Calculate the turning point and line of symmetry of $x^2+8x+17=0$

1. Complete the square of the quadratic equation

$$x^2+8x+17=0$$
$$(x+4)^2-16+17=0$$
$$(x+4)^2+1=0$$

2. Compare with general form $a(x+p)^2+q=0$ to find the values of p and q.

$$p=4, \qquad q=1$$

3. Find the turning point and line of symmetry.

The turning point is $(-p,q)=(\boldsymbol{-4},\boldsymbol{1})$
The line of symmetry is at $x=-p$ so it is at $\boldsymbol{x=-4}$.

Solving an equation

When a quadratic equation cannot be easily factorised into two brackets, an **alternative method** to solve the equation for x is to complete the square. By completing the square, the equation can be **rearranged to make x the subject**, hence **finding the solutions** to where the quadratic crosses the x axis.

Example: By completing the square, solve the equation $x^2-8x-24=0$

1. Comparing with the expression x^2+bx+c, label the coefficients.

$$b=-8, \qquad c=-24$$

2. Substitute values into the general equation to complete the square.

$$\left(x+\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2=(x-4)^2-24-(-4)^2=0$$
$$(x-4)^2-40=0$$

3. Rearrange to make x the subject.

$$(x-4)^2-40=0$$
$$(x-4)^2=40$$
$$(x-4)=\pm\sqrt{40}$$
$$(x-4)=\pm 2\sqrt{10}$$
$$x=4\pm 2\sqrt{10}$$

The solutions to this quadratic equation are $\boldsymbol{x=4+2\sqrt{10}}$ *and* $\boldsymbol{x=4-2\sqrt{10}}$.

Completing the Square (Higher Only) – Practice Questions

1. Complete the square for the following quadratic expressions:

   a) $x^2 + 6x + 6$

   b) $2x^2 + 12x + 18$

2. Find the turning point and line of symmetry of each of the following quadratic equations:

   a) $x^2 + 4x + 9 = 0$

   b) $4x^2 - 20x + 1 = 0$

3. By completing the square, solve the following quadratic equations:

   a) $x^2 + 6x + 3 = 0$

   b) $2x^2 - 4x - 10 = 0$

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.